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ON A GENERALIZATION OF KALUZA'S THEORY OF ELECTRICITY

BY A. EINSTEIN AND P. BERGMANN

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INTRODUCTION

So far, two fairly simple and natural attempts to connect gravitation and electricity by a unitary field theory have been made, one by Weyl, the other by Kaluza. Furthermore, there have been some attempts to represent Kaluza's theory formally so as to avoid the introduction of the fifth dimension of the physical continuum. The theory presented here differs from Kaluza's in one essential point; we ascribe physical reality to the fifth dimension whereas in Kaluza's theory this fifth dimension was introduced only in order to obtain new components of the metric tensor representing the electromagnetic field. Kaluza assumes the dependence of the field variables on the four coördinates x^1, x^2, x^3, x^4 and not on the fifth coördinate x^0 when a suitable coördinate system is chosen.

It is clear that this is due to the fact that the physical continuum is, according to our experience a four dimensional one. We shall show, however, that it is possible to assign some meaning to the fifth coördinate without contradicting the four dimensional character of the physical continuum.

In the first chapter of our paper we present Kaluza's original theory; in the second, its new generalization. This is done in order to make the reading easier.

In the appendix we simplify the derivation of the field equations by generalizing the tensor calculus for the case of tensor densities.

I. THE KALUZA THEORY[1]

1. We consider a five dimensional space (4 + 1 dimensions) with the metric

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

2. We assume that the space is "cylindrical" with respect to a definite vector field. Analytically this means: There exists a contravariant vector A^ν so that if τ is an infinitesimally small constant, then

$$\delta x^\nu = \tau \cdot A^\nu \tag{2}$$

is an infinitesimal displacement of the space points. The element must not

[1] Contained in part in O. Klein, Zeitschrift f. Phys., **37** (1926), 895-906, A. Einstein, Sitzungsberichte d. Preuss. Akademie d. Wissenschaften, Physik.-mathemat. Klasse, **1927**, 23-30.

change its length given by (1) if its endpoints are displaced according to (2). This is expressed by the equation:

(3a) $$\gamma_{\mu\nu,\sigma}A^{\sigma} + \gamma_{\mu\sigma}A^{\sigma}_{,\nu} + \gamma_{\nu\sigma}A^{\sigma}_{,\mu} = 0.$$

The invariant character of this equation can easily be shown. Putting

(3b) $$\gamma_{\mu\sigma}A^{\sigma} = A_{\mu},$$

we get

(3c) $$A_{\mu,\nu} + A_{\nu,\mu} - A^{\sigma}\{\gamma_{\mu\sigma,\nu} + \gamma_{\nu\sigma,\mu} - \gamma_{\mu\nu,\sigma}\} = 0,$$

or

(3d) $$\{A_{\mu,\nu} - A_{\sigma}\Gamma^{\sigma}_{\mu\nu}\} + \{A_{\nu,\mu} - A_{\sigma}\Gamma^{\sigma}_{\nu\mu}\} = 0,$$

where $\Gamma^{\lambda}_{\iota\kappa}$ are the Christoffel symbols of the second kind. Or, in the language of the absolute tensor calculus we have:

(3) $$A_{\mu;\nu} + A_{\nu;\mu} = 0.$$[2]

It follows from (3) that $\gamma_{\mu\nu}A^{\mu}A^{\nu}$ (that is the norm of A^{μ}) is constant along the lines to which A^{μ} is a tangent. By multiplying (3) by $A^{\mu}A^{\nu}$ we obtain:

(4) $$A^{\nu}A^{\mu}A_{\mu;\nu} + A^{\mu}A^{\nu}A_{\nu;\mu} = A^{\nu}(A^{\mu}A_{\mu})_{,\nu} = 0.$$

3. Kaluza's theory imposes another property on the vector field A^{μ}, besides the one expressed in (3): The lines to which the A^{μ} are tangents—the "A-lines" —have to be geodesics. Analytically this means: The lines defined by

(5) $$\frac{dx^{\nu}}{d\sigma} = \lambda A^{\nu}$$

(where $1/\lambda^2$ is equal to the norm of A) satisfy the equation

(6a) $$\frac{d^2x^{\nu}}{d\sigma^2} + \Gamma^{\nu}_{\alpha\beta}\frac{dx^{\alpha}}{d\sigma}\frac{dx^{\beta}}{d\sigma} = 0,$$

or, according to (5):

(6b) $$\frac{\partial \ln \lambda}{\partial x^{\alpha}} A^{\alpha}A^{\nu} + A^{\alpha}A^{\nu}_{;\alpha} = 0.$$

The first term vanishes because of (4) $\left(\frac{\partial \ln \lambda}{\partial x^{\alpha}} \cdot A^{\alpha}\right.$ is the derivative of a function of the norm of A in the direction of $\left.A\right)$. Therefore:

(6c) $$A^{\alpha}A^{\nu}_{;\alpha} = 0,$$

2 Killing's equation.

or

$$A^{\alpha}A_{\nu;\alpha} = 0. \tag{6d}$$

From (3) follows:

$$A^{\alpha}A_{\alpha;\nu} = \tfrac{1}{2}(A^{\alpha}A_{\alpha})_{,\nu} = 0. \tag{6}$$

This means that the norm of A is constant not only along the A-lines but in the whole space. We characterize, in this way, the space structure in Kaluza's theory.

REMARK: From the "cylindrical" character of the space there follows the existence of a vector A the *symmetrical* derivatives of which vanish. But there remain the *antisymmetrical* derivatives of A_μ which represent, in Kaluza's theory, the electromagnetic field.

The Special Coördinate System

We consider a four dimensional surface cutting each of the A-lines once and only once. We introduce on this surface 4 coördinates x^a ($a = 1 \cdots 4$) and assume x^0 equal zero on this surface. Through each point of the surface passes an A-line on which one of the two directions is chosen as positive; this choice is to be the same on all A-lines. As x^0-coördinate we choose the distance along an A-line calculated with the help of (1), starting from the original four dimensional surface with $x^0 = 0$, the chosen direction determining the sign of x^0.

Therefore, on the x^0-line:

$$x^0 = \int_0^{x^0} \sqrt{\gamma_{00}}\, dx^0, \tag{7a}$$

and therefore in the whole space,

$$\gamma_{00} = 1. \tag{7}$$

As the absolute value of A is always 1, we have

$$\gamma_{00}A^0A^0 = 1, \tag{8a}$$

or,

$$A^0 = 1, \tag{8}$$

since, because of the choice of our coördinate system, $A^1 = A^2 = A^3 = A^4 = 0$.

From this and (3a), expressing the condition of cylindricity, we have

$$\gamma_{\mu\nu,0} = 0. \tag{9}$$

As generally

$$A_\mu = \gamma_{\mu\nu}A^\nu,$$

we have in the special coördinate system,

$$A_m = \gamma_{0m}\,, \qquad A_0 = \gamma_{00} = 1. \tag{10}$$

None of the components of A_μ depend on x^0 because of (9). Therefore in the antisymmetrical derivatives

$$A_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$

all components bearing one index zero vanish identically.

In our special coördinate system the space structure is described by 10 functions γ_{mn} and 4 functions $\gamma_{0m}(= A_m)$, which do not depend on x^0. (Here and later the latin indices run only from 1 to 4.) This is, therefore, a four dimensional description of the space. But the variables introduced are not, from the point of view of our description, the most natural ones, as will be shown soon.

We ask: What coördinate transformations are still allowed preserving our special coördinate system?

a. The coördinates x^a have been chosen arbitrarily on the four dimensional surface (with $x^0 = 0$). Therefore the following transformation is still allowed:

$$\begin{aligned} \bar{x}^a &= \bar{x}^a(x^1 \cdots x^4), \\ \bar{x}^0 &= x^0. \end{aligned} \tag{11}$$

We call it the *Four-transformation.*

b. The four dimensional surface (with $x^0 = 0$) was chosen arbitrarily. It can be replaced by another surface without changing the x^a-coördinates corresponding to the A-lines. The appropriate transformation is:

$$\begin{aligned} \bar{x}^a &= x^a, \\ \bar{x}^0 &= x^0 + f(x^1 \cdots x^4). \end{aligned} \tag{12}$$

We call it the *Cut-transformation.*

The γ_{mn} behave, with respect to a four-transformation, as a four dimensional tensor and $A_n(= \gamma_{0n})$ as a four-vector. With respect to the cut-transformation we have

$$\bar{A}_m = A_m - \frac{\partial f}{\partial x^m}, \tag{13}$$

$$\bar{\gamma}_{mn} = \gamma_{mn} - \gamma_{m0}\frac{\partial f}{\partial x^n} - \gamma_{n0}\frac{\partial f}{\partial x^m} + \frac{\partial f}{\partial x^m}\frac{\partial f}{\partial x^n}. \tag{14}$$

Instead of the γ_{mn} which are not invariant with respect to cut-transformations (according to (14)) we introduce new functions g_{mn} by the following consideration: Let two infinitesimally close A-lines L and L' pass through two infinitesimally close points P and P'. There exists between L and L' an

extremal distance because of the existence of a metric. This distance is determined by

(15) $$g_{\mu\nu} = \gamma_{\mu\nu} - A_\mu A_\nu.$$

The $g_{\mu\nu}$ defined in this way are components of a five dimensional tensor. In the special coördinate system only those of its components which have no zero index do not vanish, i.e. the g_{mn} are its only components. The g_{mn} behave like a four dimensional tensor with respect to the four-transformation and are invariant with respect to cut-transformations.

It is, therefore, quite natural to choose g_{mn} and A_m as field variables in the special coördinate system; the g_{mn} are the components of the physical metrical tensor. The antisymmetrical derivatives of A_m, that is $A_{m,n} - A_{n,m}$, are invariant with respect to the cut-transformation, but not the components A_m themselves. This corresponds to the fact that the electromagnetic potentials are defined only up to additive terms which are gradients of an arbitrary function.

This result can be summarized: The five dimensional space structure is equivalent to a four dimensional one with a metric (g_{mn}) and a vector field (A_m), determined only up to an additive arbitrary gradient.

Kaluza's roundabout way of introducing the five dimensional continuum allows us to regard the gravitational and electromagnetic fields as a unitary space structure. This result is essential. At first sight it could be argued: Is it really a step forward to introduce a five dimensional metric and a vector for which we assume some arbitrary restrictions instead of a four dimensional metric and a four-vector? But to be just to Kaluza's theory it should be born in mind that the one really arbitrary step is taken only when the five dimensional representation of the four dimensional space is assumed. If, however, this is done, then the introduction of a five-vector is merely a consequence, necessary because of the four dimensional character of the empirical space. The representation of the electromagnetic field by a potential vector, achieved in this way, is certainly not a trivial result.

But apart from this point, the result is rather disappointing. Kaluza's aim was undoubtedly to obtain some new physical aspect for gravitation and electricity by introducing a unitary field structure. This end was, however, not achieved.

The field equations of the theory can be derived for instance with the help of a variational principle. This means a choice of an action function $\mathfrak{H}$, which must be a scalar density, i.e. $\frac{1}{\sqrt{g}} \cdot \mathfrak{H}$ must be an invariant with respect to the four- and cut-transformations. Therefore, in our special coördinate system, the invariant has to be formed of the g_{mn} and A_m as well as of such derivatives of these quantities which do not change if A_m is replaced by $A_m - \partial f/\partial x^m$. We can conclude, therefore, that A_m can enter the action function only through their antisymmetric derivatives. If the Hamiltonian $\mathfrak{H}$ has to contain only

second derivatives of the potentials g_{mn}, A_m, or expressions of the second order containing only first derivatives, then the most general action function is of the form

$$\mathfrak{H} = \sqrt{g}(R + c \cdot M), \tag{16}$$

where R is the curvature invariant and M the Maxwell invariant.

Many fruitless efforts to find a field representation of matter free from singularities based on this theory have convinced us, however that such a solution does not exist.[3]

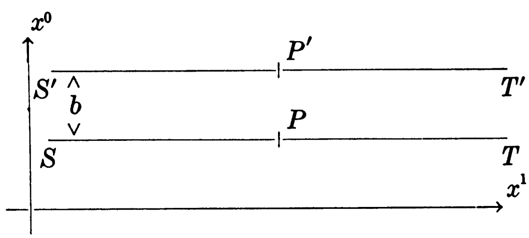
II. Generalization of Kaluza's Theory

General Remarks about the Space Structure

If Kaluza's attempt is a real step forward, then it is because of the introduction of the five dimensional space. There have been many attempts to retain the essential formal results obtained by Kaluza without sacrificing the four dimensional character of the physical space. This shows distinctly how vividly our physical intuition resists the introduction of the fifth dimension. But by considering and comparing all these attempts one must come to the conclusion that all these endeavors did not improve the situation. It seems impossible to formulate Kaluza's idea in a simple way without introducing the fifth dimension.

We have, therefore, to take the fifth dimension seriously although we are not encouraged to do so by plain experience. If, therefore, the space structure seems to force acceptance of the five dimensional space theory upon us we must ask whether it is sensible to assume the rigorous reducibility to four dimensional space. We believe that the answer should be "no," provided that it is possible to understand, in another way, the quasi-four dimensional character of the physical space by taking as a basis the five dimensional continuum and to simplify hereby the basic geometrical assumptions.

To make the underlying ideas clearer we consider a two dimensional space (instead of the five dimensional one), that is approximately a one dimensional continuum (instead of a four dimensional one).



[3] We tried to find a rigorous solution of the gravitational equations, free from singularities, by taking into account the electromagnetic field. We thought that a solution of a rotational-symmetrical character could, perhaps, represent an elementary particle. Our investigation was based on the theory of "bridges" (Einstein and Rosen, Phys. Rev., **48**: 73 (1935). We convinced ourselves, however, that no solution of this character exists.

The space considered is the narrow strip limited by the lines ST and $S'T'$, The coördinates are x^0 and x^1, and, for the sake of simplicity, the Euclidean character of the space is assumed. We imagine the strip curved into a tube so that ST coincides with $S'T'$. Every point P on ST coincides in this way with a certain point P' on $S'T'$, so that a small cylindrical surface is formed. If the width of the strip, that is, the circumference of the cylinder (denoted by b), is small, and if a continuous and slowly changing function $\varphi(x^0, x^1)$ is given, that is if $b \cdot \partial\varphi/\partial x^a$ is small compared with φ, then the values of φ belonging to the points on the segment PP' differ from each other very slightly and φ can be regarded, approximately, as a function of x^1 only. This reduction of the number of dimensions of the space was achieved because the space is closed in the x^0-dimension and the "width" is very small. But this quasi-one dimensional character of the space does not exist if the function φ varies too rapidly.

The assumption concerning the Euclidean character is certainly unessential for our consideration. It may be essential for our argument that some metric is assumed so that it is possible to speak of a circumference in the x^0-direction, but in any case we are interested only in spaces with a Riemannian metric.

It is more convenient to consider, in the following, instead of a space "closed" in the x^0-direction[4] a space "periodic" in the x^0-direction. The "periodic" and the "closed" character are equivalent if the corresponding points P, P', P'', $\cdots$ are regarded as "the same" point.

By putting in our picture the four dimensional continuum $x^1 \cdots x^4$ instead of the one dimensional continuum x^1 we obtain a model of the physical space which will form the basis of our future considerations.

The most essential point of our theory is the replacing of the hypothesis 2, of rigorous cylindricity by the assumption that space is closed (or periodic) in the x^0-direction.

The Space Structure

We shall characterize the space according to our previous remarks concerning the modified form of Kaluza's theory.

1. We consider a five dimensional space with the metric (1).

2. The space is closed with respect to one dimension. This closed space will be represented by a space which is open and periodic with respect to this dimension. A point P of the physical space will, therefore, be represented by an infinite number of points P, P', P'', $\cdots$.

REMARK: This postulate replaces the cylindricity condition in Kaluza's original theory.

3. Through every space point there passes a geodesic line closed in itself and free from singularities. Or, formulated for the "periodic" space: If two points P, P' corresponding to each other, but otherwise arbitrary, are connected by

[4] This expression is not quite clear but it may be used here in a chapter aiming only at a general orientation.

the geodesic line,[5] then this line also passes through all other points corresponding to P, i.e. through P'', P''' etc.

This axiom is analogous to the axiom 3. of the original Kaluza theory. The geodesic lines introduced here correspond to the A-lines in Kaluza's theory and we shall again call them the A-lines.

We again introduce the special coördinate system. We consider a four dimensional surface $x^0 = 0$ cutting all A-lines once and only once. The four coördinates $x^1 \cdots x^4$ on this surface determine a point on this surface as well as an A-line through this point. Any point P on this A-line is determined by the distance (measured along this line with the help of (1)) between $\mathring{P}$ (that is the intersecting point of the A-line and the surface $x^0 = 0$) and P. For points on the "positive" side of the surface $x^0 = 0$ we have

$$x^0 = \frac{1}{b} \int_{\mathring{P}}^{P} d\sigma, \tag{17}$$

where b is

$$b = \int^{P'} d\sigma. \tag{17a}$$

Here P, P' are two corresponding, consecutive points lying on the A-line. Therefore, b depends on $x^1 \cdots x^4$ only. By this convention the difference of the x^0-coördinates of two corresponding, consecutive points (which, of course, lie on the same x^0-line) equals 1.

By the assumptions thus made five coördinates x^0, $x^1 \cdots x^4$ correspond to every space point.

Lines with constant $x^1 \cdots x^4$ are geodesic lines. On these lines we have:

$$\frac{dx^1}{d\sigma} = \frac{dx^2}{d\sigma} = \frac{dx^3}{d\sigma} = \frac{dx^4}{d\sigma} = 0, \qquad \frac{dx^0}{d\sigma} = \frac{1}{b}, \qquad \frac{d^2x^0}{d\sigma^2} = 0. \tag{18a}$$

Therefore, the equation of the geodesic line gives

$$\Gamma^{\alpha}_{00} = 0. \tag{18b}$$

From this follows:

$$\begin{bmatrix} \alpha \\ 00 \end{bmatrix} = 0, \tag{18c}$$

or

$$\gamma_{00,0} = 0 \tag{18d}$$

and

$$2\gamma_{a0,0} - \gamma_{00,a} = 0. \tag{18}$$

[5] We do not consider the existence of more than one geodesic connection.

Equation (18d) does not contain anything new because we have on the A-lines:

(19a) $$d\sigma^2 = b^2 dx^{02}$$

and

(19) $$\gamma_{00} = b^2$$

which is independent of x^0.

We integrate (18) over one period along the A-line. The integral of the first term on the left hand side in (18) gives zero because of the periodic character of the space. Because of (17) and (19) we obtain:

(20) $$\frac{\partial}{\partial x^a} \int_{P'}^{P''} \gamma_{00} dx^0 = \frac{\partial}{\partial x^a} \left\{ b \int_{P'}^{P''} d\sigma \right\} = \frac{\partial}{\partial x^a} (b^2) = 0.$$

Therefore b and γ_{00} are independent of $x^1 \cdots x^4$ too, i.e. constant in the whole space. We may write:

(21) $$\begin{cases} \gamma_{00} = 1, \\ b = 1. \end{cases}$$

In a strict sense we were not justified in choosing the value for γ_{00} as we have already chosen the coördinate difference Δx^0 of corresponding, consecutive points to be equal 1. We take this fact into account by putting Δx^0 not 1, but some small quantity λ. Then, because of (21), λ is simultaneously the metric distance $P'P''$.

Furthermore, from (18) follows because of (21)

(22) $$\frac{\partial}{\partial x^0} \gamma_{a0} = 0.$$

Therefore, in the generalized theory also, the γ_{0a} (but not the γ_{ab}) are independent of x^0. We can introduce, in the generalized theory as well as before, the field of contravariant unit vectors A^μ the components of which are, in the special coördinate system, 0, 0, 0, 0, 1. Therefore again

(22a) $$A_\alpha = A^\beta \gamma_{\beta\alpha} = \gamma_{0\alpha}.$$

We can also introduce as the metric tensor instead of the

(23) $$g_{\alpha\beta} = \gamma_{\alpha\beta} - A_\alpha A_\beta,$$

of which only those components which have no index zero do not vanish. There exists, in this sense, here also a four dimensional metrical tensor g_{ab}. *Its components are, however, in general periodic functions of* x^0. The whole difference between the new theory and Kaluza's lies just in this fact. (A_m in both theories do not depend on x^0.)

Having characterized the space structure we shall turn later, after some mathematical preparations, to the formulation of field equations which can be

derived from a variational principle in the space considered. We shall again demand that the Hamiltonian be composed additively of terms containing either second derivatives (linearly) or products of two first derivatives.

Tensor Analysis with Respect to the Special Coördinate System

From our definition of the special coördinate system it follows that again there exists the freedom of "four-transformations" and "cut-transformations." In order to avoid the formal difficulties of general coördinate systems we have to investigate the transformation properties of the quantities formed with respect to the four- and cut-transformations. Definition of the contravariant four-vector: Four quantities a^s (being functions of $x^0, x^1 \cdots x^4$) form a contravariant vector if, with respect to four-transformations, the following transformation law is valid:

$$\bar{a}^s = \frac{\partial \bar{x}^s}{\partial x^t} a^t, \tag{24}$$

and if the a^s are invariant with respect to cut-transformations. The definition of a covariant vector is quite analogous. The definition of a tensor also follows from this.

Therefore g_{mn} is a tensor. However $A_m (= \gamma_{0m})$ is not a vector as the components transform, with respect to a cut-transformation, according to (13). The $A_{mn} = A_{m,n} - A_{n,m}$ are, however components of a tensor.

The tensor algebra is the same as the normal four dimensional one and does not need to be carried out here.

We now direct our attention to the formation of new tensors by differentiation.

1. For both four- and cut-transformations we have

$$\frac{\partial}{\partial \bar{x}^0} = \frac{\partial}{\partial x^0}. \tag{25}$$

Therefore: From every tensor a new one may be formed by differentiation with respect to x^0.

2. Covariant differentiation with respect to x^a: If ψ is a scalar then we have for cut-transformations:

$$\frac{\partial \psi}{\partial \bar{x}^a} = \frac{\partial \psi}{\partial x^a} - \frac{\partial \psi}{\partial x^0} f_{,a}. \tag{26a}$$

From this equation and (13), by eliminating $f_{,a}$, we get that

$$\frac{\partial \psi}{\partial x^a} - A_a \frac{\partial \psi}{\partial x^0}$$

is invariant with respect to the cut-transformations. Therefore the operator

$$\left(\frac{\partial}{\partial x^a} - A_a \frac{\partial}{\partial x^0}\right), \tag{26}$$

acting on a tensor, leaves its components invariant with respect to cut-transformations.

On the other hand, we know from the absolute differential calculus that the expression

$$B_{s,a} - \tfrac{1}{2}B_l g^{lm}(g_{ms,a} + g_{am,s} - g_{as,m}) \tag{27a}$$

(B_s being an arbitrary covariant vector) has a tensor character with respect to four- (but not cut-) transformations. By replacing $\dfrac{\partial}{\partial x^a}$ in this expression everywhere by the operator (26) we add new members which again have a tensor character with respect to four-transformations, this being true for $B_{s,0}$, $g_{mn,0}$, A_s. The expression obtained in this way certainly has a tensor character with respect to four-transformations. Since it is also invariant with respect to cut-transformations it satisfies our definition of a tensor.

We can, therefore, define an invariant differentiation in the following way:

$$B_{s;a} = B_{s,a} - A_a B_{s,0} - B_l \Gamma^l_{sa}, \tag{27}$$

where

$$\Gamma^l_{sa} = \tfrac{1}{2}g^{lm}[(g_{ms,a} - A_a g_{ms,0}) + (g_{ma,s} - A_s g_{ma,0}) - (g_{as,m} - A_m g_{as,0})]. \tag{27b}$$

In an analogous way we have for a contravariant vector with the same Γ^l_{sa}:

$$B^s_{;a} = B^s_{,a} - A_a B^s_{,0} + B^l \Gamma^s_{la}. \tag{28}$$

From this a corresponding rule for more general tensors follows immediately. The absolute differentiation defined thus satisfies the rule of differentiation of products.

By straightforward calculation one obtains

$$g_{mn;a} = g^{mn}{}_{;a} = 0. \tag{29}$$

3. The curvature tensor: By a straightforward calculation we find:

$$B_{s;a;b} - B_{s;b;a} = -B_l R^l_{.sab} - B_{s,0} A_{ab}, \tag{30}$$ [6]

where

$$R^l_{.sab} = (\Gamma^l_{sa,b} - A_b \Gamma^l_{sa,0}) - (\Gamma^l_{sb,a} - A_a \Gamma^l_{sb,0}) - \Gamma^l_{am}\Gamma^m_{sb} + \Gamma^l_{bm}\Gamma^m_{sa}. \tag{30a}$$

From (30) follows, because of the tensor character of the last term, that $R^l_{.sab}$ is a tensor (the curvature tensor) which is antisymmetric with respect to its last two indices.

4. The commutation rule for the operators "$_{,0}$" and "$_{;a}$" is

$$(u_{m,0})_{;a} - (u_{m;a})_{,0} = u_s \Gamma^s_{am,0}. \tag{31}$$

The derivatives of the Γ^s_{am} with respect to x^0 have, therefore, tensor character

[6] $A_{ab} = A_{a,b} - A_{b,a}$.

Variational Principle and Field Equations

Starting from the formalism developed here we shall formulate the most general field equations satisfying the following conditions:

1. The field equations should be derived from a variational principle;
2. The action function should consist exclusively of terms containing either second derivatives linearly or products of two derivatives of the first order.

These are the tensors from which such an action function could be built algebraically:

$$R^{i}_{.klm}\,,\ A_{mn}\,,\ g_{mn}\,,\ g_{mn,0}\,,\ g_{mn,00}\,,\ A_{mn,0}\,,\ A_{mn;s}.$$

Some of the invariants formed from these tensors can, by partial integration of the Hamiltonian integral be reduced to the others. There remain only the following invariants which are, in this sense, independent of each other:

$$(32)\quad \begin{cases} \underset{1}{H} = R^{i}_{.klm}\delta^{l}_{i}g^{km} = R_{km}g^{km} = R, \qquad \underset{2}{H} = A_{mn}A_{st}g^{ms}g^{nt} = A_{mn}A^{mn}, \\ \underset{3}{H} = g^{mn}{}_{,0}g_{mn,0}, \qquad \underset{4}{H} = g^{mn}g_{mn,0}g^{rs}g_{rs,0}. \end{cases}$$

The most general variational principle, satisfying our conditions formulated above, is therefore:

$$(33)\quad \delta\int \mathfrak{H}dx^0dx^1dx^2dx^3dx^4 = 0, \qquad \mathfrak{H} = \sqrt{-g}\left(\underset{1}{\alpha}\underset{1}{H} + \underset{2}{\alpha}\underset{2}{H} + \underset{3}{\alpha}\underset{3}{H} + \underset{4}{\alpha}\underset{4}{H}\right),$$

where $\underset{1}{\alpha} \cdots \underset{4}{\alpha}$ are arbitrary constants. The variation has to be carried out with respect to the g^{mn} and the A_s. The integral should be taken over an arbitrary world region of the coördinates $x^1 \cdots x^4$ and over one period of the coördinates x^0. The latter restriction is necessary because the δA_s must be chosen independent of x^0. (Compare (22)). Therefore the contributions from the boundary of the region do not enter into the usual operations performed on the action function only if the integral is taken over exactly one period of x^0. The calculations, which may be performed with the help of the methods quoted in the appendix, give:

$$(34)\quad \begin{cases} \delta\int \underset{1}{\mathfrak{H}}dx^0 \cdots dx^4 = \int \{[\tfrac{1}{2}R_{kl} + \tfrac{1}{2}R_{lk} - \tfrac{1}{2}g_{kl}R]\delta g^{kl} \\ \qquad\qquad + [g^{mn}\Gamma^{s}_{mn,0} - g^{ms}\Gamma^{n}_{mn,0}]\delta A_s\}\sqrt{-g}\,dx^0 \cdots dx^4, \\ \delta\int \underset{2}{\mathfrak{H}}dx^0 \cdots dx^4 = \int \{[2A_{km}A_{l.}^{\ m} - \tfrac{1}{2}g_{kl}A_{mn}A^{mn}]\delta g^{kl} \\ \qquad\qquad - [4A^{st}{}_{;t}]\delta A_s\}\sqrt{-g}\,dx^0 \cdots dx^4 \\ \delta\int \underset{3}{\mathfrak{H}}dx^0 \cdots dx^4 = \int [-2g_{kl,00} + 2g^{rs}g_{kr,0}g_{ls,0} - g^{rs}g_{rs,0}g_{kl,0} \\ \qquad\qquad - \tfrac{1}{2}g^{rs}{}_{,0}g_{rs,0}g_{kl}]\delta g^{kl}\sqrt{-g}\,dx^0 \cdots dx^4, \\ \delta\int \underset{4}{\mathfrak{H}}dx^0 \cdots dx^4 = \int g_{kl}[\tfrac{1}{2}(g^{mn}g_{mn,0})^2 + 2g^{mn}g_{mn,00} \\ \qquad\qquad + 2g^{mn}{}_{,0}g_{mn,0}]\delta g^{kl}\sqrt{-g}\,dx^0 \cdots dx^4. \end{cases}$$

Here the δg^{kl} are arbitrary functions of the coördinates $x^0 \cdots x^4$. The sum of the coefficients of δg^{kl}, multiplied by their $\underset{i}{\alpha}$, must, therefore, vanish. However, the δA_s are, as we mentioned before, functions of $x^1 \cdots x^4$ only, but independent of x^0. Therefore, in this case, only the integral of the sum of the coefficients of δA_s, (multiplied by their $\underset{i}{\alpha}$), taken over one period of x^0, vanishes. The fourteen field equations derived in this way from the variational principle (33), take the form:

$$(35)\quad \begin{cases} \underset{1}{\alpha}(\tfrac{1}{2}R_{kl} + \tfrac{1}{2}R_{lk} - \tfrac{1}{2}g_{kl}R) + \underset{2}{\alpha}(2A_{km}A_{l.}^{m} - \tfrac{1}{2}g_{kl}A_{mn}A^{mn}) + \\ + \underset{3}{\alpha}(-2g_{kl,00} + 2g^{rs}g_{kr,0}g_{ls,0} - g^{rs}g_{rs,0}g_{kl,0} - \tfrac{1}{2}g^{rs}{}_{,0}g_{rs,0}g_{kl}) + \\ + \underset{4}{\alpha}g_{kl}[\tfrac{1}{2}(g^{mn}g_{mn,0})^2 + 2g^{mn}g_{mn,00} + 2g^{mn}{}_{,0}g_{mn,0}] = 0 = \dfrac{1}{\sqrt{-g}}\mathfrak{G}_{kl}, \end{cases}$$

$$(36)\quad \int \{\underset{1}{\alpha}(g^{mn}\Gamma^{s}_{mn,0} - g^{ms}\Gamma^{n}_{mn,0}) - 4\underset{2}{\alpha}A^{st}_{;t}\}\sqrt{-g}\,dx^0 = 0 = \int \mathfrak{J}^s\,dx^0.$$

Finally we shall derive the identities which the field equations satisfy. As it is customary, we shall find these identities by expressing analytically the invariance of the Hamiltonian integral with respect to an infinitesimal coördinate transformation. If we introduce for δg^{mn} and δA_s the variations produced by an infinitesimal coördinate transformation, then the variation of the integral vanishes. It should be noted that the variation is not to be carried out with the space points fixed, but with the coördinate values fixed. Denoting the infinitesimal coördinate transformations by

$$(37a)\quad \begin{cases} x^a = \bar{x}^a + \xi^a(\bar{x}^1 \cdots \bar{x}^4) \\ x^0 = \bar{x}^0 + \xi^0(\bar{x}^1 \cdots \bar{x}^4) \end{cases}$$

we obtain for δg^{mn} and δA_s :

$$(37b)\quad \begin{cases} \delta g^{kl} = -g^{sl}\xi^k_{,s} - g^{sk}\xi^l_{,s} + g^{kl}{}_{,s}\xi^s + g^{kl}{}_{,0}\xi^0, \\ \delta A_s = A_r\xi^r_{,s} + \xi^0_{,s} + A_{s,r}\xi^r. \end{cases}$$

These expressions can easily be brought into a form revealing more clearly their tensor character:

$$(37c)\quad \begin{cases} \delta g^{kl} = -g^{sl}\xi^k{}_{;s} - g^{sk}\xi^l_{;s} + g^{kl}{}_{,0}(A_r\xi^r + \xi^0), \\ \delta A_s = (A_r\xi^r + \xi^0)_{;s} + A_{sr}\xi^r. \end{cases}$$

The expression $(A_r\xi^r + \xi^0)$ is an inner product of two five dimensional vectors, therefore a scalar. After performing some partial integrations (using the methods explained in the appendix) we obtain:

$$(37d)\quad \delta\int \mathfrak{H}dx^\alpha = \int \{[\mathfrak{G}_{k.}{}^{s}{}_{;s} + A_{sk}\mathfrak{J}^s]\xi^k + [\mathfrak{G}_{rs}g^{rs}{}_{,0} - \mathfrak{J}^s_{;s}](A_r\xi^r + \xi^0)\}\,dx^\alpha = 0.$$

As ξ^k, ξ^0 are arbitrary functions of $x^1 \cdots x^4$, but independent of x^0 the integral (37d) will vanish identically if and only if the integrals of the square bracket expressions over one period of x^0 equal zero:

$$\left\{\begin{aligned} & 2\int \mathfrak{G}_{k.\,;s}^{\;\,s}\, dx^0 + A_{sk} \int \mathfrak{J}^s\, dx^0 = 0, \\ & \int \mathfrak{G}_{rs} g^{rs}{}_{,0}\, dx^0 - \left\{ \int \mathfrak{J}^s\, dx^0 \right\}_{;s} = 0. \end{aligned}\right. \tag{37}$$

These are the identities of the field equations (35), (36).

The theory involves four universal constants, $\underset{2}{\alpha}/\underset{1}{\alpha}$, $\underset{3}{\alpha}/\underset{1}{\alpha}$, $\underset{4}{\alpha}/\underset{1}{\alpha}$, λ. The constant $\underset{2}{\alpha}/\underset{1}{\alpha}$ corresponds to the gravitational constant involving a connection between the (arbitrary) units of length and mass. One of the other constants, for instance λ, will depend on the unit of length. The remaining two are "genuine" universal constants which cannot be eliminated from the theory.

Summary

Kaluza's five dimensional theory of the physical space provides a unitary representation of gravitation and electromagnetism. It demands, if consistently interpreted as a five dimensional theory, the existence of a vector A, whose symmetrical covariant derivatives vanish in the whole space. Apart from its physical meaning, this assumption seems to be artificial. Furthermore the representation of a purely four dimensional continuum by a five dimensional one appears not to be sufficiently justified.

By our generalization we try to eliminate these objections. The hypothesis of cylindricity, (i.e. of the existence of a vector A with vanishing symmetrical derivatives) is replaced by the assumption that the space is closed in the direction of the fifth coördinate. By this change the basic assumptions of the theory are considerably simplified. Furthermore it is much more satisfactory to introduce the fifth dimension not only formally, but to assign to it some physical meaning. Nevertheless there is no contradiction with the empirical four dimensional character of physical space.

Appendix

Analysis of Tensor Densities

The tensor density plays, beside the concept of tensors, an important rôle in relativistic investigations. Although the concept of tensor densities can easily be reduced to that of tensors nevertheless the whole formalism and especially the derivation of the field equations from the variational principle are considerably simplified if tensor densities are introduced independently. Although the analytical properties of tensor densities have been fully investi-

gated[7] we shall, in order to make the reading easier, make some remarks on the subject.

A tensor density differs from a tensor by a factor (common to all components) having the transformation character of a scalar density. A scalar density $\underset{(n)}{\rho}$ of the "weight" n is a quantity with the following transformation property:

$$\underset{(n)}{\bar{\rho}} = \underset{(n)}{\rho} \left| \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \right|^n . \tag{A 1}$$

There exist, correspondingly, tensor densities of different weights. (Ordinary tensors are tensor densities of the weight zero.) By multiplying two tensor densities of the weights m and n we obtain a tensor density of the weight $(m + n)$. The rules of a tensor density algebra follow immediately from that of the tensor algebra and need not be derived here.

For the sake of simplicity we shall consider here only quantities in a four dimensional continuum; the results are, however, independent of the number of dimensions.

There exist, in every four dimensional space, the two tensor densities of Levi-Civita

$$\underset{(-1)}{\delta_{iklm}}$$

and

$$\underset{(+1)}{\delta_{iklm}},$$

antisymmetric in all indices, the components of which are $+1$ or -1, depending of whether $(iklm)$ is an odd or an even permutation of (1234). Their tensor density character follows immediately from the transformation law of tensor densities. From the Levi Civita tensor densities the well known Kronecker tensor $\delta_m^{m'}$ (with components equal 1 if $m = m'$ and 0 if $m \neq m'$) can be obtained by:

$$\delta_m^{m'} = \tfrac{1}{6} \underset{(-1)}{\delta_{iklm}} \underset{(+1)}{\delta^{iklm'}} . \tag{A 2}$$

We assume in the space under consideration the existence of symmetrical components of parallel displacement (the space functions $\Gamma^l_{ik} = \Gamma^l_{ki}$) by which the covariant derivatives of a tensor are defined in the usual way:

$$B^{k\cdots}_{i\cdots;s} = B^{k\cdots}_{i\cdots,s} + B^{l\cdots}_{i\cdots}\Gamma^k_{ls} + \cdots - B^{k\cdots}_{l\cdots}\Gamma^l_{is} - \cdots . \tag{A 3}$$

This definition of the covariant differentiation has two essential properties:

1. It satisfies the rule for the differentiation of products of the ordinary calculus:

$$(AB)_{;s} = AB_{;s} + BA_{;s} .$$

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[7] V. Hlavatý, Annalii di Mathematica, V, Ser. IV (1927/28): *Théorie des densités dans le displacement général.*

2. The absolute derivatives of the Kronecker tensor vanish identically:

$$\delta^k_{i;s} = 0 .$$

We shall now define the covariant derivative of a tensor density so that:

1. The rule for the differentiation of products is preserved;
2. The absolute derivatives of the co- and contravariant tensor densities of Levi Civita vanish identically.

By direct calculation we confirm that these demands are satisfied by the following definition:

$$\text{(A 4)} \qquad \underset{(n)}{B^{k\cdots}_{i\cdots;s}} = \underset{(n)}{B^{k\cdots}_{i\cdots,s}} + \underset{(n)}{B^{l\cdots}_{i\cdots}}\Gamma^k_{ls} + \cdots - \underset{(n)}{B^{k\cdots}_{l\cdots}}\Gamma^l_{is} - \cdots - n\underset{(n)}{B^{k\cdots}_{i\cdots}}\Gamma^r_{sr}.$$

This is the rule for the differentiation of tensor densities of the weight n.

If we specialize the space further by assuming that it is Riemannian then the components of the parallel displacement are given by:

$$\text{(A 5)} \qquad \Gamma^s_{mn} = \tfrac{1}{2}g^{st}(g_{ml,n} + g_{nl,m} - g_{mn,l}),$$

and therefore

$$\text{(A 6)} \qquad \Gamma^n_{sn} = \frac{1}{2}\frac{g_{,s}}{g}, \qquad g = |\,g_{ik}\,|.$$

The determinant g is a scalar density of the weight 2. By applying (A 4) we get:

$$\text{(A 7)} \qquad \underset{(2)}{g_{;s}} = g_{,s} - 2g\left(\frac{1}{2}\frac{g_{,s}}{g}\right) = 0.$$

Therefore the absolute derivative of $\sqrt{g}$ and, because of the rule for the differentiation of products, the absolute derivatives of $g_{mn}\sqrt{g}$ and of $g^{mn}\sqrt{g}$ vanish also.

We shall now show how the tensor densities can be applied in the variational process.

1. The derivation of the equations of gravitation by the variation of Riemann's curvature tensor:

$$\text{(A 8)} \qquad \begin{cases} \delta \displaystyle\int \underset{(1)}{\mathfrak{H}}\, d\tau = 0, \\ \underset{(1)}{\mathfrak{H}} = \sqrt{-g}\,g^{km}\delta^l_i R^i_{.klm}. \end{cases}$$

It was shown by Palatini that the variation with respect to the g^{km} can be simplified, because of the equation:

$$\text{(A 9)} \qquad \delta R^i_{.klm} = (\delta\Gamma^i_{kl})_{;m} - (\delta\Gamma^i_{km})_{;l}$$

which can easily be verified (note that $\delta\Gamma^i_{kl}$ is a tensor!). Because of

$$\text{(A 10a)} \qquad \delta(\sqrt{-g}\,g^{km}) = \sqrt{-g}(\delta g^{km} - \tfrac{1}{2}g^{km}g_{ab}\cdot\delta g^{ab}),$$

we have then

$$\text{(A 10b)}\quad \delta \int \underset{(1)}{\mathfrak{H}}\, d\tau = \int \sqrt{-g}(R_{km} - \tfrac{1}{2}g_{km}R)\delta g^{km}\, d\tau + \int \sqrt{-g}\, g^{km}[(\delta\Gamma^l_{kl})_{;m} - (\delta\Gamma^l_{km})_{;l}]\, d\tau.$$

As $(g^{km}\cdot\sqrt{-g})_{;l}$ vanishes the integrand of the second integral may be written

$$\text{(A 10c)}\quad [\sqrt{-g}(g^{kl}\delta\Gamma^m_{km} - g^{km}\delta\Gamma^l_{km})]_{;l},$$

where the square bracket expression $\underset{(1)}{K^l}$ is a vector density of the weight 1. It follows, therefore, from (A 4) that $\underset{(1)}{K^l_{;l}}$ may be replaced by $\underset{(1)}{K^l_{,l}}$. This integral vanishes, according to Gauss' theorem, if the variations of the g_{ik} and of the Γ^l_{ik} vanish on the boundary of the integration region. There remains only the first integral of (A 10b) so that the result is

$$\text{(A10)}\quad R_{km} - \tfrac{1}{2}g_{km}R = 0.$$

2. Tensor densities in the space of the generalized Kaluza theory. In such a space, and in the special coördinate system, a scalar density of the weight n can be defined by the transformation law:

$$\text{(A 11)}\quad \begin{aligned} \underset{(n)}{\bar{\rho}} &= \underset{(n)}{\rho}\left|\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right|^n && \text{for four-transformations,} \\ \underset{(n)}{\bar{\rho}} &= \underset{(n)}{\rho} && \text{for cut-transformations.} \end{aligned}$$

The transformation law of tensor densities is also determined by this definition, i.e. reduced to that of tensors.

Covariant differentiation of tensor densities: We can expect, because of (A 4) and (27), (28), that the following formula expresses the differentiation rule:

$$\text{(A 12)}\quad \underset{(n)}{B^{k\cdots}_{i\cdots;s}} = \underset{(n)}{B^{k\cdots}_{i\cdots,s}} - A_s \underset{(n)}{B^{k\cdots}_{i\cdots,0}} + \underset{(n)}{B^{l\cdots}_{i\cdots}}\Gamma^k_{ls} + \cdots - \underset{(n)}{B^{k\cdots}_{l\cdots}}\Gamma^l_{is} - \cdots - n\, B^{k\cdots}_{i\cdots}\Gamma^l_{sl}.$$

$\underset{(-1)}{\delta_{iklm}}$ is a tensor density with respect to four-transformations and invariant with respect to cut-transformations. It is, therefore, a tensor density according to our definitions. The absolute derivative of this tensor density, given by (A 12), vanishes. It can also be seen that (A 12) preserves the rule for the differentation of products.

It follows from our definitions that $\underset{(2)}{g} = |g_{mn}|$ is a scalar density of the weight 2 with the absolute derivatives:

$$\text{(A 13)}\quad \underset{(2)}{g_{,s}} = g_{,s} - A_s g_{,0} - 2g\Gamma^l_{sl} = 0.$$

(This vanishes because of (27b).) Furthermore the equation holds:

(A 14) $$g_{mn;s} = 0.$$

We form the divergence $\underset{(1)}{B}{}^{s}_{;s}$ of a vector density $\underset{(1)}{B}{}^{s}$ of the weight 1. According to (A 12) we have

(A 15a) $$\underset{(1)}{B}{}^{s}_{;s} = \underset{(1)}{B}{}^{s}_{,s} - A_s \underset{(1)}{B}{}^{s}_{,0},$$

and as A_s does not depend on x^0

(A 15) $$\underset{(1)}{B}{}^{s}_{;s} = \underset{(1)}{B}{}^{s}_{,s} - (A_s \underset{(1)}{B}{}^{s})_{,0}.$$

It is essential to note here that both terms on the right hand side take the form of ordinary derivatives.

We have now, as in the gravitational theory, to form the variation of the integral (Compare (32), (33))

(A 16a) $$\int \sqrt{-g}\, g^{km} \delta^l_i R^i_{.klm}\, d\tau.$$

This integral, however, is to be taken over a region of the variables $x^0, x^1 \cdots x^4$, which contains exactly one period of the coördinate x^0 (and is arbitrary with respect to $x^1 \cdots x^4$). The $R^i_{.klm}$ are defined by (30a). The variation of this quantity takes the form:

(A 16b) $$\delta R^i_{.klm} = (\delta \Gamma^i_{kl})_{,m} - (\delta \Gamma^i_{km})_{;l} - \delta A_m \Gamma^i_{kl,0} + \delta A_l \Gamma^i_{km,0}.$$

The variation of this integral gives, as far as the variation of the factor $g^{km} \cdot \sqrt{-g}$ is concerned, the contribution:

(A 16c) $$\int \sqrt{-g}(\tfrac{1}{2}R_{km} + \tfrac{1}{2}R_{mk} - \tfrac{1}{2}g_{km}R)\delta g^{km}\, d\tau.$$

The variation of $R^i_{.klm}$ gives, because of the two last terms in (A 16b),

(A 16d) $$\int \sqrt{-g}(g^{km}\Gamma^l_{km,0} - g^{kl}\,\Gamma^m_{km,0})\delta A_l\, d\tau.$$

The first two terms on the right hand side of (A 16b), however, do not give any contribution. The proof is similar for both terms; we shall, therefore, show it only for the first one. The integrand

$$\sqrt{-g} \cdot g^{km}(\delta \Gamma^l_{kl})_{;m}$$

can, because of

$$(\sqrt{-g} \cdot g^{km})_{;m} = 0,$$

(following from (A 13), (A 14)) be brought into the form:

$$\underset{(1)}{B}{}^{m}_{;m}.$$

But such an integral vanishes because of (A 15). It does so partially because the variations vanish on the x^a-boundaries, partially because of the periodicity of B^m with respect to x^0. The final result of the variation is:

$$\text{(A 16)} \quad \int \sqrt{-g}\{(\tfrac{1}{2}R_{km} + \tfrac{1}{2}R_{mk} - \tfrac{1}{2}g_{km}R)\delta g^{km} + (g^{km}\Gamma^l_{km,0} - g^{kl}\Gamma^m_{km,0})\delta A_l\}\, d\tau,$$

where the δg^{km} depend arbitrarily on x^a, x^0, the δA_s however only on x^a.

The calculations applied in the derivation of the identities are quite similar.

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